

SERIES REPRESENTATION OF TIME-STABLE STOCHASTIC PROCESSES

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Abstract. A stochastically continuous process $\xi(t)$, $t \geq 0$, is said to be *time-stable* if the sum of n i.i.d. copies of ξ equals in distribution the time-scaled stochastic process $\xi(nt)$, $t \geq 0$. The paper advances the understanding of time-stable processes by means of their LePage series representations as the sum of i.i.d. processes with the arguments scaled by the sequence of successive points of the unit intensity Poisson process on $[0, \infty)$. These series yield numerous examples of stochastic processes that share one-dimensional distributions with a Lévy process.

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1. INTRODUCTION

The (strict) stability property of stochastic processes is conventionally defined by requiring that the sum of i.i.d. copies of a process is distributed as the space-scaled variant of the original process. An alternative scaling operation applied to the time argument leads to another definition of stability.

DEFINITION 1.1. A stochastically continuous real-valued process $\xi(t)$, $t \geq 0$, is said to be *time-stable* if, for each $n \geq 2$,

$$(1.1) \quad \xi_1 + \dots + \xi_n \stackrel{\mathcal{D}}{\sim} n \circ \xi,$$

where ξ_1, \dots, ξ_n are i.i.d. copies of ξ , $\stackrel{\mathcal{D}}{\sim}$ means the equality of all finite-dimensional distributions and $(n \circ \xi)(t) = \xi(nt)$, $t \geq 0$, is the process obtained by time scaling ξ .

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Definition 1.1 goes back to Mansuy [18], where processes satisfying (1.1), regardless of their stochastic continuity, are called *infinitely divisible with respect to time* (IDT), see also [4], Section 6.7. Indeed, they are infinitely divisible in the sense that ξ can be represented as the sum of n i.i.d. processes for each $n \geq 2$. However, the “time-stable” name better emphasises the particular stability feature of such processes. These processes have recently been investigated in [8], [12], also with a multivariate time argument. Time-stable processes with values in \mathbb{R}^d can be defined similarly to Definition 1.1. Similarly to other stable random elements, time-stable processes naturally appear as limits for time-scaled sums of stochastic processes.

The major difficulty in the analysis of time-stable processes stems from the necessity to work with the whole paths of the processes. The time-stability concept cannot be formulated in terms of finite-dimensional distributions at any given time moments, since the time argument on the right-hand side of (1.1) is scaled.

Definition 1.1 can be modified to introduce α -time-stable processes as

$$\xi_1 + \dots + \xi_n \stackrel{\mathcal{D}}{\sim} n^{1/\alpha} \circ \xi,$$

where each $\alpha \neq 0$ is possible. This concept appears in Example 8.12 of [7] as an example of the stability property in the cone of continuous functions with scaling applied to the argument. While such processes (for general α) have been considered in [11], the process $\xi(t^{1/\alpha})$, $t \geq 0$, obtained by time change is time-stable (with $\alpha = 1$) and so it is not necessary to study α -time stability for general $\alpha \neq 1$.

Another closely related concept is that of a *dilatively stable process* ζ that satisfies the following scaling relation for some $\alpha > 0$, $\delta \in (0, 2\alpha]$, and all $n \geq 2$:

$$\zeta_1 + \dots + \zeta_n \stackrel{\mathcal{D}}{\sim} n^{1/2-\alpha/\delta} (n^{1/\delta} \circ \zeta),$$

see [14], where such processes are also assumed to have moments of all orders and the left-hand side is replaced by the convolution power for finite-dimensional distributions of order n with not necessarily integer n (which however does not alter the family of processes). If ζ is dilatively stable, then $\xi(t) = t^{1/2-\alpha/\delta} \zeta(t^{1/\delta})$, $t \geq 0$, satisfies (1.1) and so is a time-stable process if ζ is stochastically continuous.

Barczy et al. [1] extended the setting from [14] by allowing α and δ be arbitrary real numbers and relaxing the moment conditions. They also defined (ρ_1, ρ_2) -*aggregate self-similar processes* ζ for arbitrary real numbers ρ_1 and ρ_2 by the scaling relation

$$\zeta_1 + \dots + \zeta_n \stackrel{\mathcal{D}}{\sim} n^{\rho_1} (n^{-\rho_2} \circ \zeta),$$

so that for $\rho_1 = \rho_2$ one recovers the aggregate similar process from [16]. It is easy to see that $t^{\rho_1} \zeta(t^{-\rho_2})$, $t \geq 0$, satisfies (1.1), so that this and all other above-mentioned generalisations may be obtained by time and scale change from time-stable processes. An exponential time change leads to translatively stable processes, see [13], Definition 2.4.3. A similar concept was introduced by Penrose [21], who

called a non-negative stochastic process ξ *semi-min-stable* if $\min(\xi_1(t), \dots, \xi_n(t))$ shares the finite-dimensional distributions with $n^{-1}\xi(n^\alpha t)$, $t \geq 0$.

Section 2 discusses elementary properties of time-stable processes. The infinite divisibility of such processes makes it possible to use their spectral representation obtained in [15] and then show that the Lévy measure is homogeneous with respect to time scaling, see Section 3. The main result of Section 4 and of the whole paper is the LePage representation of time-stable processes whose Lévy measures are supported by the family of right-continuous functions with left limits. In particular, this is the case for non-negative processes. The obtained LePage representation yields the series representations for dilatively stable and aggregate self-similar processes. The structure of pure jump time-stable processes is closely related to the stability property of marked point processes; in this case the LePage representation is similar to the cluster representation of infinitely divisible point processes, see Section 5.

The concept of time stability allows generalisations in various directions. The necessary structure consists of a time set which is invariant under scaling by arbitrary positive real numbers and an associative and commutative binary operation which is applied pointwisely to the values of processes. For instance, the definition applies also to stochastic processes defined on the whole line and on \mathbb{R}^d or with addition replaced by the maximum operation.

While (1.1) actually defines a *strictly* time-stable stochastic process, the stability concept can be relaxed by replacing the right-hand side with $n \circ \xi + f_n$ for deterministic functions $\{f_n\}$. Moreover, it is possible to consider random measures stable with respect to scaling of their argument (see [7], Example 8.23) and also time-stable generalised stochastic processes, i.e. random generalised functions.

2. ELEMENTARY PROPERTIES

The following standard result provides an alternative definition of time-stable processes.

PROPOSITION 2.1. *A stochastically continuous process $\xi(t)$, $t \geq 0$, is time-stable if and only if*

$$(2.1) \quad a \circ \xi_1 + b \circ \xi_2 \stackrel{\mathcal{D}}{\sim} (a + b) \circ \xi$$

for all $a, b > 0$, where ξ_1 and ξ_2 are independent copies of ξ .

Each Lévy process is time-stable, see [4], Section 6.7. If ξ is time-stable, then there exists a unique Lévy process $\tilde{\xi}$, called the *associated Lévy process* of ξ , such that $\tilde{\xi}(t)$ coincides in distribution with $\xi(t)$ for each given $t \geq 0$, see [18], Proposition 4.1. Thus, the characteristic function of $\xi(t)$ is given by

$$(2.2) \quad \mathbf{E} \exp\{\imath \lambda \xi(t)\} = \exp\{-t \Psi(\lambda)\}, \quad t \geq 0, \lambda \in \mathbb{R},$$

where Ψ denotes the *cumulant* of $\tilde{\xi}$ and also of ξ .

It follows from (2.2) that $\xi(t)$ weakly converges to zero as $t \downarrow 0$, which corresponds to the stochastic continuity of ξ , since $\xi(0) = 0$ a.s. by (1.1). Furthermore, if $\xi(t)$ and $\xi(s)$ share the same distribution for $t \neq s$, then ξ is a.s. zero.

Comparing the one-dimensional distributions shows that if the non-degenerate time-stable process is a.s. non-negative for any $t > 0$, then it is a.s. non-negative everywhere, its one-dimensional distributions are increasing in the stochastic order, and $\sup_{t \geq 0} \xi(t)$ is a.s. infinite. In contrast to Lévy processes, non-negative time-stable processes need not be a.s. monotone, for example, $\xi(t) = N(2t) - N(t)$, $t \geq 0$, if N is the standard Poisson process.

THEOREM 2.1. *A time-stable process ξ is identically distributed as the sum of a linear function, a centred Gaussian process with covariance function C that satisfies $C(ut, us) = uC(t, s)$ for all $t, s, u \geq 0$, and an independent time-stable process without Gaussian component.*

Proof. Since ξ is infinitely divisible, its finite-dimensional distributions are infinitely divisible. The rest follows by comparing the Lévy triplets of the n -fold convolution of $(\xi(t_1), \dots, \xi(t_k))$ and of $(\xi(nt_1), \dots, \xi(nt_k))$ for any $t_1, \dots, t_k \geq 0$ and $k, n \geq 1$. ■

Various characterisations of Gaussian time-stable processes are presented in [18]. In the following we only consider time-stable processes without a Gaussian part.

3. LÉVY MEASURES OF TIME-STABLE PROCESSES

Each stochastically continuous process is separable in probability (also is said to satisfy Condition S from [25], Definition 3.11.2), meaning the existence of an at most countable set $T_0 \subset [0, \infty)$ such that for all $t \geq 0$, there exists a sequence $t_n \in T_0$, $n \geq 1$, such that $\xi(t_n)$ converges to $\xi(t)$ in probability. The spectral representation of infinitely divisible stochastic processes that are separable in probability and do not have a Gaussian component is obtained in [15], Theorem 2.14, using a Poisson process on a certain space (Ω, \mathfrak{F}) with a σ -finite measure μ . Reformulating this result for (Ω, \mathfrak{F}) being the space $\mathbb{R}^{[0, \infty)}$ with the cylindrical σ -algebra \mathfrak{C} , we see that an infinitely divisible stochastically continuous process ξ without a Gaussian component admits a spectral representation

$$(3.1) \quad \xi(t) \stackrel{\mathcal{D}}{\sim} c(t) + \int_{\mathbb{R}^{[0, \infty)} \setminus \{0\}} f(t) d\Pi_Q(f),$$

where c is a deterministic function and $\Pi_Q = \{f_i(\cdot) : i \geq 1\}$ is the Poisson process on $\mathbb{R}^{[0, \infty)} \setminus \{0\}$ with a σ -finite intensity measure Q such that

$$(3.2) \quad \int_{\mathbb{R}^{[0, \infty)} \setminus \{0\}} \min(1, f(t)^2) Q(df) < \infty$$

for all $t \geq 0$. The measure Q is called the *Lévy measure* of ξ . The integral with respect to Π_Q in (3.1) is defined as the a.s. existing limit of the compensated sums

$$(3.3) \quad \lim_{r \downarrow 0} \left[\sum_{f_i \in \Pi_Q} f_i(t) \mathbf{1}_{|f_i(t)| > r} - \int_{\{f: |f(t)| > r\}} L(f(t)) Q(df) \right],$$

where

$$(3.4) \quad L(u) = \begin{cases} u, & |u| \leq 1, \\ 1, & u > 1, \\ -1, & u < -1, \end{cases}$$

is a Lévy function, see also [19].

Furthermore, Theorem 2.14 in [15] ensures the existence of a *minimal* spectral representation, meaning that the σ -algebra generated by $\{f : f(t) \in A\}$ for all $t \geq 0$ and Borel $A \subset \mathbb{R}$ coincides with the cylindrical σ -algebra \mathfrak{C} on $\mathbb{R}^{[0, \infty)}$ up to Q -null sets and there is no set $B \in \mathfrak{C}$ with $Q(B) > 0$ such that for every $t \geq 0$, $Q(\{f \in B : f(t) \neq 0\}) = 0$. In the following assume that the cylindrical σ -algebra \mathfrak{C} is complete with respect to Q . By [15], Theorem 2.17, the minimal spectral representation is unique up to an isomorphism, and so the Lévy measure is well defined.

The stochastic continuity of ξ implies that ξ has a measurable modification, see [10], Theorem 3.3.1. Then Proposition 2.19 in [15] establishes that the representation (3.1) involves a measurable function $c(t)$, $t \geq 0$, and that the functions f from Π_Q can be chosen to be strongly separable. The latter means the existence of a measurable null-set $\Omega_0 \subset \mathbb{R}^{[0, \infty)}$ and a countable set $\mathbb{Q} \subset [0, \infty)$ (called a *separant*) such that, for each open $G \subset [0, \infty)$ and closed $F \subset \mathbb{R}$, we have

$$(3.5) \quad \{f : f(t) \in F \forall t \in G \cap \mathbb{Q}\} \setminus \{f : f(t) \in F \forall t \in G\} \subset \Omega_0.$$

If (3.2) is strengthened to require

$$(3.6) \quad \int_{\mathbb{R}^{[0, \infty)} \setminus \{0\}} \min(1, |f(t)|) Q(df) < \infty,$$

then the integral (3.1) is well defined without taking the limit and without the compensating term in (3.3), so that

$$(3.7) \quad \xi(t) \stackrel{\mathcal{D}}{\sim} c(t) + \sum_{f_i \in \Pi_Q} f_i(t)$$

for a deterministic function c . It is well known that (3.6) holds if $\xi(t)$ is a.s. non-negative for all $t \geq 0$, see e.g. [26], Theorem 51.1.

LEMMA 3.1. *For each $B \in \mathfrak{C}$ and $s > 0$, the set $s \circ B = \{s \circ f : f \in B\}$ also belongs to \mathfrak{C} .*

Proof. If B is a cylinder, then $s \circ B \in \mathfrak{C}$, and the statement follows from the monotone class argument. ■

The next result follows from the fact that $\xi(0) = 0$ a.s. for a time-stable process ξ .

LEMMA 3.2. *The Lévy measure of a time-stable process is supported by $\{f \in \mathbb{R}^{[0,\infty)} \setminus \{0\} : f(0) = 0\}$.*

LEMMA 3.3. *An infinitely divisible stochastically continuous process ξ without a Gaussian component is time-stable if and only if $c(t) = bt$, $t \geq 0$, for a constant $b \in \mathbb{R}$ and the Lévy measure Q satisfies*

$$(3.8) \quad Q(s \circ B) = s^{-1}Q(B), \quad s > 0,$$

for all $B \in \mathfrak{C}$.

Proof. The sufficiency follows from the expression for the characteristic function of the finite-dimensional distributions of ξ ,

$$(3.9) \quad \mathbf{E} \exp \left\{ i \sum_j \theta_j \xi(t_j) \right\} \\ = \exp \left\{ i b \sum_j \theta_j t_j + \int \left[\exp \left(i \sum_j \theta_j f(t_j) \right) - 1 - i \sum_j \theta_j L(f(t_j)) \right] Q(df) \right\}.$$

Now assume that ξ is time-stable. Comparing the characteristic functions of the finite-dimensional distributions for the processes on the left- and right-hand side of (2.1) and using the uniqueness of the Lévy triplets show that the function c is additive and so is linear in view of its measurability.

The Lévy measure corresponding to the minimal spectral representation of the process on the left-hand side of (2.1) is $Q(a^{-1} \circ B) + Q(b^{-1} \circ B)$. In view of the uniqueness of the minimal spectral representation (see [15], Theorem 2.17), the Lévy measures of the processes on the left- and right-hand side of (2.1) coincide. Thus

$$Q(a^{-1} \circ B) + Q(b^{-1} \circ B) = Q((a+b)^{-1} \circ B)$$

for all $a, b > 0$ and all $B \in \mathfrak{C}$. Since Q is non-negative, Theorem 1.1.7 in [3] implies that $Q(s^{-1} \circ B)$ is a linear function of s , i.e. (3.8) holds. ■

The same scaling property for the Lévy measure appears in [18], Lemma 5.1, and later on was reproduced in [11], Proposition 4.1, for time-stable processes with paths in the Skorokhod space of right-continuous functions with left limits (*càdlàg* functions). The proof there is however incomplete, since it is not shown that the Lévy measure of such a process is supported by *càdlàg* functions.

PROPOSITION 3.1. *If $\xi(t)$, $t \geq 0$, is a time-stable càdlàg process with a.s. non-negative values, then its Lévy measure Q is supported by càdlàg functions.*

Proof. In this case the Lévy measure Q satisfies (3.6) and so ξ admits the representation (3.7). If ξ' is an independent copy of ξ , then $\xi - \xi'$ is symmetric and has the series decomposition with the Lévy measure supported by càdlàg (free of oscillatory discontinuities) functions, see [22], Theorem 4. The support of Q is a subset of the support of the Lévy measure for $\xi - \xi'$. ■

4. LEPAGE SERIES REPRESENTATION

In finite-dimensional spaces, Lévy measures of strictly stable laws admit a polar decomposition into the product of a radial and a finite-directional part, and the corresponding sum (if necessary compensated) of points of the Poisson process is known as the LePage series, see [25], Corollary 3.10.4, and [17], [23]. The LePage series can be defined in general spaces [7], where they provide a rich source of stable laws and in many cases characterise stable laws.

The following result provides the LePage series characterisation for time-stable processes without a Gaussian part and whose Lévy measure is supported by the family \mathbb{D}' of not identically vanishing càdlàg functions on $[0, \infty)$. We endow the family \mathbb{D}' with the σ -algebra induced by \mathfrak{C} . Let \mathbb{D}'_0 be the family of not identically vanishing càdlàg functions that vanish at the origin.

THEOREM 4.1. *The following statements are equivalent for a stochastically continuous càdlàg process $\xi(t)$, $t \geq 0$.*

(i) *The process ξ is time-stable without a Gaussian part and with its Lévy measure Q supported by \mathbb{D}' .*

(ii) *The stochastic process ξ is infinitely divisible without a Gaussian part, with a deterministic linear part, its Lévy measure Q is supported by \mathbb{D}'_0 , and*

$$(4.1) \quad Q(B) = \int_0^\infty \sigma(t \circ B) dt$$

for each measurable $B \subset \mathbb{D}'_0$ and a probability measure σ on \mathbb{D}'_0 such that

$$(4.2) \quad \int_{\mathbb{D}'_0} \int_0^\infty \min(1, f(t)^2) t^{-2} dt \sigma(df) < \infty.$$

(iii) *The stochastic process ξ has the same distribution as*

$$(4.3) \quad bt + \lim_{r \downarrow 0} \left[\sum_{i=1}^\infty \varepsilon_i(\Gamma_i^{-1}t) \mathbf{1}_{|\varepsilon_i(\Gamma_i^{-1}t)| > r} - \mathbf{E} \int_0^\infty L(\varepsilon(s^{-1}t)) \mathbf{1}_{|\varepsilon(s^{-1}t)| > r} ds \right], \quad t \geq 0,$$

where the limit exists almost surely, $b \in \mathbb{R}$ is a constant, L is defined as in (3.4), $\{\varepsilon_i, i \geq 1\}$ is a sequence of i.i.d. stochastic processes distributed as ε , such that ε a.s. takes values in \mathbb{D}'_0 ,

$$(4.4) \quad \mathbf{E} \int_0^\infty \min(1, \varepsilon(t)^2) t^{-2} dt < \infty,$$

and $\{\Gamma_i, i \geq 1\}$ is the sequence of successive points of a homogeneous unit intensity Poisson process on $[0, \infty)$.

Proof. By Lemma 3.3, a time-stable process without a Gaussian part can be alternatively described as an infinitely divisible stochastically continuous process whose Lévy measure Q satisfies (3.8) and so is supported by \mathbb{D}'_0 . It is obvious that Q given by (4.1) satisfies (3.8). It remains to show that the scaling property (3.8) yields (4.1), so that (i) implies (ii).

The following construction is motivated by the argument used to prove Theorem 10.3 in [9]. By Lemma 3.2, Q is supported by \mathbb{D}'_0 . Decompose \mathbb{D}'_0 into the union of disjoint sets

$$\mathbb{X}_0 = \{f : \sup_{t \geq 0} |f(t)| > 1\}$$

and

$$\mathbb{X}_k = \{f : \sup_{t \geq 0} |f(t)| \in (2^{-k}, 2^{-k+1}], f \notin \mathbb{X}_j, j = 0, \dots, k-1\}, \quad k \geq 1.$$

In view of the completeness assumption on the σ -algebra, all sets \mathbb{X}_k , $k \geq 0$, are measurable. Recall the separant \mathbb{Q} and the exceptional set Ω_0 from (3.5) that holds due to the assumed stochastic continuity and infinite divisibility of ξ . Denote by $\tilde{\mathbb{X}}_k$, $k \geq 0$, the analogues of \mathbb{X}_k where the supremum is taken over the set of non-negative rational numbers. Since

$$\mathbb{X}_0^c = \{f : |f(t)| \leq 1, t \in [0, \infty)\},$$

we have $\mathbb{X}_0 \setminus \tilde{\mathbb{X}}_0 \subset \Omega_0$. Similarly, $\mathbb{X}_k \setminus \tilde{\mathbb{X}}_k \subset \Omega_0$ for all $k \geq 1$.

For each $k \geq 0$, define the map $\tau_k : \mathbb{X}_k \rightarrow (0, \infty)$ by

$$\tau_k(f) = \inf\{t > 0 : |f(t)| > 2^{-k}\}, \quad f \in \mathbb{X}_k.$$

Since all functions from \mathbb{D}'_0 vanish at the origin, $\tau_k(f)$ is strictly positive and finite, and $\tau_k(c \circ f) = c^{-1} \tau_k(f)$ for all $c > 0$. Let

$$\mathbb{S}_k = \{f \in \mathbb{X}_k : \tau_k(f) = 1\}.$$

Then $|f(1)| \geq 2^{-k}$ for all $f \in \mathbb{S}_k$, $k \geq 0$, and each function $g \in \mathbb{X}_k$ can be uniquely represented as $s \circ f$ for $f \in \mathbb{S}_k$ and $s > 0$. The maps $(f, s) \mapsto s \circ f$ and $g \mapsto$

$(\tau_k(g) \circ g, \tau_k(g)^{-1})$ are mutually inverse measurable bijections between the sets $\mathbb{S}_k \times (0, \infty)$ and \mathbb{X}_k . This is seen by using the separability assumption (3.5) and Lemma 3.1. The right-continuity of f and (3.5) imply that

$$\Delta_k(f) = \sup\{t \in \mathbb{Q} : |f(s)| \geq 2^{-k-1} \text{ for all } s \in [1, 1+t]\}, \quad f \in \mathbb{S}_k,$$

is strictly positive and Borel measurable for each $k \geq 0$. Define

$$\begin{aligned} \mathbb{S}_{k0} &= \{f \in \mathbb{S}_k : \Delta_k(f) > 1\}, \\ \mathbb{S}_{kj} &= \{f \in \mathbb{S}_k : \Delta_k(f) \in (2^{-j}, 2^{-j+1}]\}, \quad j \geq 1. \end{aligned}$$

Then \mathbb{S}_k is the disjoint union of \mathbb{S}_{kj} for $j \geq 0$ and \mathbb{X}_k is the disjoint union of

$$\mathbb{X}_{kj} = \{s \circ f : f \in \mathbb{S}_{kj}, s > 0\}, \quad j \geq 0.$$

Fix any $k, j \geq 0$. Then

$$\begin{aligned} q_{kj} &= Q(\{s \circ f : f \in \mathbb{S}_{kj}, s \in [1, 1+2^{-j}]\}) \\ &\leq Q(\{f \in \mathbb{D}'_0 : |f(1)| \geq 2^{-k-1}\}) \\ &\leq 2^{2k+2} \int_{\{f: |f(1)| \geq 2^{-k-1}\}} \min(1, f(1)^2) Q(df) \\ &\leq 2^{2k+2} \int \min(1, f(1)^2) Q(df) < \infty. \end{aligned}$$

By (3.8),

$$\begin{aligned} &Q(\{s \circ f : f \in \mathbb{S}_{kj}, s \geq 1\}) \\ &\leq \sum_{i=0}^{\infty} Q(\{s \circ f : f \in \mathbb{S}_{kj}, s \in [(1+2^{-j})^i, (1+2^{-j})^{i+1}]\}) \\ &= \sum_{i=0}^{\infty} (1+2^{-j})^{-i} q_{kj} < \infty. \end{aligned}$$

Thus, Q restricted onto \mathbb{X}_{kj} is a push-forward under the map $(f, s) \rightarrow s \circ f$ of the product $\eta_{kj} \otimes \theta$ of a finite measure η_{kj} supported by \mathbb{S}_{kj} and the measure θ on $(0, \infty)$ with density $s^{-2}ds$. Let c_{kj} be some positive number; then the measure σ_{kj} defined on \mathbb{D}'_0 by $\sigma_{kj}(B) = c_{kj}\eta_{kj}(c_{kj}^{-1} \circ B)$ assigns all its mass to the set $c_{kj} \circ \mathbb{S}_{kj}$. Then the push-forward of $\sigma_{kj} \otimes \theta$ under the map $(f, s) \rightarrow s \circ f$ is Q restricted on \mathbb{X}_{kj} and the total mass of σ_{kj} equals $c_{kj}\eta_{kj}(\mathbb{S}_{kj})$. By choosing c_{kj} appropriately, it is always possible to achieve that $\sigma = \sum_{k,j \geq 0} \sigma_{kj}$ is a probability measure on \mathbb{D}'_0 . Combining the push-forward representations of Q restricted to \mathbb{X}_{kj} , $k, j \geq 0$, we see that Q is the push-forward of $\sigma \otimes \theta$ and so (4.1) holds. Given (4.1), (4.2) is equivalent to (3.2).

The equivalence of (ii) and (iii) is immediate by choosing ε to be i.i.d. with distribution σ and noticing that (4.2) is equivalent to (4.4) and that the limit in (4.3) corresponds to the limit in (3.3). Note that $\{\Gamma_i^{-1}, i \geq 1\}$ form the Poisson process on \mathbb{R}_+ with intensity $s^{-2}ds$. ■

REMARK 4.1. *There are many probability measures σ that satisfy (4.1), and so the distribution of ε in (4.3) is not unique. For example, it is possible to scale the time arguments of $\{\varepsilon_i, i \geq 1\}$ by a sequence of i.i.d. positive random variables of mean one. The distribution of ε is unique if ε is supported by a given measurable set $S' \subset \mathbb{D}'_0$ such that each $f \in \mathbb{D}'_0$ can be uniquely represented as $c \circ g$ for $c > 0$ and $g \in S'$.*

REMARK 4.2. *It follows from Theorem 3.1 in [2] that the LePage series (4.3) converges uniformly for t from any compact subset of $(0, \infty)$. If $H(t, r, V) = \varepsilon(t/r)$, then Condition (3.3) of [2] becomes*

$$\int_0^\infty \mathbf{P}\{(\varepsilon(t_1/r), \dots, \varepsilon(t_k/r)) \in B\} dr = Q\left(\{f : (f(t_1), \dots, f(t_k)) \in B\}\right)$$

for all Borel B in $\mathbb{R}^k \setminus \{0\}$, $t_1, \dots, t_k \geq 0$, and $k \geq 1$.

THEOREM 4.2. *A stochastically continuous stochastic process ξ is time-stable without a Gaussian part and with the Lévy measure Q supported by \mathbb{D}' and satisfying (3.6) if and only if*

$$(4.5) \quad \xi(t) \stackrel{\mathcal{D}}{\sim} bt + \sum_{i=1}^\infty \varepsilon_i(\Gamma_i^{-1}t), \quad t \geq 0,$$

where the series converges almost surely, $b \in \mathbb{R}$ is a constant, $\{\varepsilon_i, i \geq 1\}$ is a sequence of i.i.d. stochastic processes with realisations in \mathbb{D}'_0 such that

$$(4.6) \quad \mathbf{E} \int_0^\infty \min(1, |\varepsilon(t)|) t^{-2} dt < \infty,$$

and $\{\Gamma_i, i \geq 1\}$ is the sequence of successive points of the homogeneous unit intensity Poisson process on $[0, \infty)$.

PROOF. It suffices to note that (4.6) is equivalent to (3.6). ■

COROLLARY 4.1. *Each a.s. non-negative càdlàg time-stable process admits the LePage representation (4.5).*

REMARK 4.3. *Condition (4.6) (respectively (4.4)) holds if $\int_0^1 \mathbf{E}|\varepsilon(t)|t^{-2}dt < \infty$ (respectively $\int_0^1 \mathbf{E}(\varepsilon(t)^2)t^{-2}dt < \infty$). For example, this is the case if $\varepsilon(t) = 0$, $t \in [0, \tau]$, for a positive random variable τ such that τ^{-1} is integrable.*

REMARK 4.4. *Analogues of the above results hold for time-stable processes with values in \mathbb{R}^d . This can be shown by replacing S_{kj} from the proof of Theorem 4.1 with the Cartesian product of d -tuples of such sets $S_{k_1j_1}^1 \times \dots \times S_{k_dj_d}^d$, $k_i, j_i \geq 0$, $i = 1, \dots, d$, constructed for each of the coordinates of the process. In particular, Corollary 4.1 applies for time-stable processes with values in \mathbb{R}_+^d .*

EXAMPLE 4.1 (*Lévy processes*). The spectral representation (3.1) of a Lévy process ξ without a Gaussian part can be obtained by setting $f_i(t) = m_i \mathbf{1}_{t \geq \tau_i}$, where $\{(\tau_i, m_i), i \geq 1\}$ is a marked Poisson process on $(0, \infty) \times (\mathbb{R} \setminus \{0\})$ with intensity measure being the product of the Lebesgue measure and a Lévy measure Λ on $\mathbb{R} \setminus \{0\}$. Indeed, then

$$\xi(t) \stackrel{\mathcal{D}}{\sim} bt + \lim_{r \downarrow 0} \left[\sum_{|m_i| > r} m_i \mathbf{1}_{\tau_i \leq t} - t \int_{|x| > r} L(x) \Lambda(dx) \right],$$

which is the classical decomposition of a Lévy process. In view of the uniqueness of the minimal spectral representation, the Lévy measure Q is supported by step functions of type $m \mathbf{1}_{t \geq \tau}$. By Theorem 4.1, ξ admits the series decomposition (4.3) with $\varepsilon(t) = \eta \mathbf{1}_{t \geq 1}$, where (4.4) corresponds to $\mathbf{E}[\min(1, \eta^2)\zeta] < \infty$. Following the construction from the proof of Theorem 4.1, the joint distribution of (η, ζ) can be constructed as follows. Write $B_0 = \{x \in \mathbb{R} : |x| > 1\}$ and $B_k = \{x \in \mathbb{R} : 2^{-k} < |x| \leq 2^{-k+1}\}$, $k \geq 1$, let $q_k = \Lambda(B_k)$, $k \geq 0$, and choose strictly positive $\{c_k, k \geq 0\}$ such that $\sum_{k=0}^{\infty} c_k q_k = 1$. Then

$$\mathbf{P}\{\eta \in A, \zeta = c_k^{-1}\} = \Lambda(A \cap B_k) c_k$$

for every Borel $A \subset \mathbb{R} \setminus \{0\}$ and $k \geq 0$. It is easy to see that

$$\mathbf{E}[\min(1, \eta^2)\zeta] = \int_{\mathbb{R} \setminus \{0\}} \min(1, x^2) \Lambda(dx).$$

If ξ has bounded variation, then Theorem 4.2 applies and

$$\xi(t) \stackrel{\mathcal{D}}{\sim} bt + \sum_{i=1}^{\infty} \eta_i \mathbf{1}_{t \geq \Gamma_i}$$

provides a LePage representation of ξ on the whole \mathbb{R}_+ , cf. [24] for the LePage representation of Lévy processes on $[0, 1]$. The choice of $\varepsilon(t) = \eta \mathbf{1}_{t \geq 1}$, $t \geq 0$, yields the compound Poisson process $\xi(t)$, which becomes the standard Poisson process if $\eta = 1$ a.s.

The time and the size of the jump of ε may be dependent. For instance, let $\varepsilon(t) = \eta \mathbf{1}_{t \geq \eta}$ for a positive random variable η . This random function always satisfies (4.6) and yields the Lévy process

$$\xi(t) = \sum_{i=1}^{\infty} \eta_i \mathbf{1}_{t \geq \Gamma_i \eta_i}$$

with the cumulant $\Psi(\lambda) = \mathbf{E}[(1 - e^{i\lambda\eta})\eta^{-1}]$.

EXAMPLE 4.2. If $\varepsilon(t) = \eta t^{1/\alpha}$, where $\alpha \in (0, 2)$ and η is a symmetric random variable with $\mathbf{E}|\eta|^\alpha < \infty$, then the LePage series (4.5) converges a.s., by Theorem 1.4.2 of [25], to $\xi(t) = bt + \zeta t^{1/\alpha}$ for a symmetric α -stable random variable

ζ , see [18]. If $\alpha = 1$ and $b = 0$, then $\xi(t) = \zeta t$ for the Cauchy random variable ζ . This yields a time-stable process with stationary increments, which is not a Lévy process. If $\alpha < 1$, the symmetry of η is not required for the convergence of the LePage series and ζ is strictly α -stable by Theorem 1.4.5 in [25].

EXAMPLE 4.3. Choosing ε to be a stochastic process with stationary increments yields examples of time-stable processes with stationary increments which are not Lévy processes. For instance, let ε be the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Then (4.4) holds since

$$\mathbf{E} \int_0^1 \min(1, \varepsilon(t)^2) t^{-2} dt \leq \int_0^1 \mathbf{E} \varepsilon(t)^2 t^{-2} dt = \int_0^1 t^{2H-2} dt < \infty.$$

EXAMPLE 4.4 (*Sub-stable processes*). Let $\varepsilon(t) = \xi(t^{1/\alpha})$, $t \geq 0$, for $\alpha \in (0, 1)$ and a time-stable process ξ such that $\mathbf{E}|\xi(1)| < \infty$. Then (4.6) holds since

$$\int_0^1 \mathbf{E}|\xi(t^{1/\alpha})| t^{-2} dt = \mathbf{E}|\xi(1)| \int_0^1 t^{1/\alpha-2} dt < \infty.$$

By conditioning on $\{\Gamma_i\}$ and using Proposition 2.1, one obtains

$$\sum_{i=1}^{\infty} \varepsilon_i(\Gamma_i^{-1}t) = \sum_{i=1}^{\infty} \xi_i(\Gamma_i^{-1/\alpha} t^{1/\alpha}) \stackrel{\mathcal{D}}{\sim} \xi(t^{1/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha}) = \xi(t^{1/\alpha} \zeta)$$

for a strictly α -stable non-negative random variable ζ independent of ξ . Then the LePage series (4.5) yields the process $X(t) = \xi(t^{1/\alpha} \zeta)$, $t \geq 0$, where ξ is time-stable and ζ is a positive strictly α -stable random variable independent of ξ , with $\alpha \in (0, 1)$. The process X is called *sub-stable* in view of the construction of sub-stable random elements in [25], Section 1.3.

EXAMPLE 4.5 (*Subordination by time-stable processes*). Let ξ be a non-decreasing time-stable process that admits the LePage representation (4.5) with $b = 0$. If $\{X_i, i \geq 1\}$ are i.i.d. copies of a Lévy process X independent of ξ , then

$$\sum_{i=1}^{\infty} X_i(\varepsilon_i(\Gamma_i^{-1}t))$$

is the LePage representation of the time-stable process $X(\xi(t))$. This is seen by conditioning upon ε_i and $\{\Gamma_i, i \geq 1\}$ and noticing that X is stochastically continuous. The time-stability property of $X(\xi(t))$ is proved in [8], Theorem 3.6, directly by computing the characteristic function.

EXAMPLE 4.6 (*Random convex broken lines*). Consider $\varepsilon(t) = (t - 1)_+$, i.e. the positive part of $(t - 1)$. Then the graph of ξ is the continuous convex broken

line with vertices at $(0, 0)$ and at

$$(\Gamma_n, \Gamma_n \sum_{i=1}^n \Gamma_i^{-1} - n), \quad n \geq 1.$$

In order to obtain a differentiable curve, it is possible to use $\varepsilon(t) = (t - 1)_+^\beta$ for $\beta > 1$.

5. TIME-STABLE STEP FUNCTIONS

Assume that ξ is a *pure jump* time-stable process, i.e. its paths are *càdlàg* piecewise constant functions with finitely many jumps in each finite interval in $[0, \infty)$ and a.s. vanishing at zero. In view of the assumed stochastic continuity and Lemma 1.6.2 of [27], the jump times of ξ have non-atomic distributions. The jump part of any *càdlàg* time-stable process is also time-stable by noticing that the jump part of the sum of two independent stochastic processes with non-atomic distribution of jump times is equal to the sum of their jump parts. This also applies to the process of jumps larger than $\delta > 0$ in absolute value.

PROPOSITION 5.1. *The time of the first jump of a non-degenerate càdlàg pure-jump time-stable process has an exponential distribution.*

PROOF. Observe that the time of the first jump of the sum of n independent processes equals the minimum of the first jump times τ_1, \dots, τ_n of all summands. Then (1.1) implies that $n^{-1}\tau$ has the same distribution as the minimum of n i.i.d. copies of τ and so characterises the exponential distribution. ■

The time of the second jump is not necessarily distributed as the sum of two independent exponential random variables since the times between jumps may be dependent and the waiting time between the first and the second jump is no longer exponentially distributed in general.

Let $\mathcal{M}((0, \infty) \times \mathbb{R})$ denote the family of marked point configurations on $(0, \infty)$ with marks from \mathbb{R} . A *marked point process* is a random element in the product space $\mathcal{M}((0, \infty) \times \mathbb{R})$, see [5], Section 6.4. The successive ordered jump times $\{\tau_k\}$ and the jump heights $\{m_k\}$ of a pure jump time-stable process ξ form the marked point process $M = \{(\tau_k, m_k), k \geq 1\}$, so that

$$\xi(t) = \sum_{\tau_k \leq t} m_k, \quad t \geq 0.$$

The sum is finite for every t since the process is assumed to have only a finite number of jumps in any bounded interval. This construction introduces a correspondence between pure jump processes and marked point processes. Note that M is a random closed (and locally finite) set in $(0, \infty) \times \mathbb{R}$, see [20]. The process

ξ is compound Poisson if and only if M is an independently marked homogeneous Poisson process, i.e. the jump times form a homogeneous Poisson process on $(0, \infty)$, while the jump sizes are i.i.d. random variables independent of the jump times.

Scaling the argument of a pure jump process ξ can be rephrased in terms of scaling the marked point process M corresponding to ξ , so that $a \circ \xi$ corresponds to the marked point process

$$a^{-1} \circ M = \{(a^{-1}\tau_k, m_k) : k \geq 1\}.$$

The sum of independent pure jump processes corresponds to the superposition of the corresponding marked point processes. The next result relates the time-stability property to the union stability of random sets (see [20], Section 4.1.3); it immediately follows from (1.1).

PROPOSITION 5.2. *A stochastically continuous pure jump process ξ is time-stable if and only if its corresponding marked point process M is a union-stable random closed set in the sense that*

$$(5.1) \quad M_1 \cup \dots \cup M_n \stackrel{\mathcal{D}}{\sim} n^{-1} \circ M$$

for each $n \geq 2$, where M_1, \dots, M_n are independent copies of M .

COROLLARY 5.1. *A stochastically continuous pure jump process ξ is time-stable if and only if $\xi = \xi_+ - \xi_-$ for the pair of stochastically continuous pure jump processes (ξ_+, ξ_-) that form a pure jump time-stable process with values in \mathbb{R}_+^2 .*

Proof. For $(\tau, m) \in (0, \infty) \times \mathbb{R}$, let $f(\tau, m) = (\tau, m_+, m_-)$, with m_+ and m_- being the positive and negative parts of $m \in \mathbb{R}$. Then M satisfies (5.1) if and only if $f(M)$ satisfies the analogue of (5.1) with the scaling along the first coordinate. Finally, this property of $f(M)$ is a reformulation of the time stability of (ξ_+, ξ_-) , where ξ_+ is the sum of all positive jumps of ξ and ξ_- is the sum of all negative jumps. ■

THEOREM 5.1. *A stochastically continuous pure jump process ξ is time-stable if and only if*

$$(5.2) \quad \xi(t) \stackrel{\mathcal{D}}{\sim} \sum_{i=1}^{\infty} \varepsilon_i(\Gamma_i^{-1}t), \quad t \geq 0,$$

where $\{\Gamma_i, i \geq 1\}$ form a homogeneous unit intensity Poisson point process on $(0, \infty)$, and $\{\varepsilon_i, i \geq 1\}$ are independent copies of a random step function ε defined on $[0, \infty)$ which is independent of $\{\Gamma_i\}$ and satisfies (4.6).

Proof. The sufficiency is immediate and follows from Theorem 4.2. For the necessity, consider the map f and the random set M from the proof of Corollary 5.1 and note that $f(M)$ is an infinitely divisible point process on $(0, \infty) \times \mathbb{R}_+^2$. It is well known (see e.g. [6], Theorem 10.2.V) that such infinitely divisible marked point process can be represented as a superposition of point configurations that build a Poisson point process on $\mathcal{M}((0, \infty) \times \mathbb{R}_+^2)$. The unique intensity measure \tilde{Q} of this Poisson process is called the KLM measure of M . This measure can also be viewed as the Lévy measure, see [7], Corollary 6.9.

Each point configuration from $\mathcal{M}((0, \infty) \times \mathbb{R}_+^2)$ corresponds to a pure jump function. The push-forward of \tilde{Q} under this correspondence is the Lévy measure of (ξ_+, ξ_-) that is supported by pure jump (and so *càdlàg*) functions. Since the components of (ξ_+, ξ_-) are non-negative, Remark 4.4 yields its representation as

$$(\xi_+(t), \xi_-(t)) \stackrel{\mathcal{P}}{\sim} \sum_{i=1}^{\infty} (\varepsilon'_i(\Gamma_i^{-1}t), \varepsilon''_i(\Gamma_i^{-1}t)),$$

so that ξ admits the series representation (5.2) with $\varepsilon = \varepsilon' - \varepsilon''$. ■

REMARK 5.1. *In the classical LePage series for random vectors, it is possible to scale the directional component to bring its norm to one. However, it is not possible in general to rescale the argument of $\{\varepsilon_i, i \geq 1\}$ from (5.2) in order to ensure that each function has the first jump at time one.*

REMARK 5.2. *It is possible to derive Theorem 5.1 from the LePage representation of the marked point process M as the union of clusters corresponding to the Poisson cluster process determined by \tilde{Q} . The corresponding series representation then becomes*

$$M = \bigcup_{i=1}^{\infty} \Gamma_i \circ E_i,$$

where $\{E_i, i \geq 1\}$ is a point process on $\mathcal{M}((0, \infty) \times \mathbb{R}_+^2)$ with the intensity measure \tilde{Q} .

If ε has a single jump only, then (5.2) yields a Lévy process, see Example 4.1.

EXAMPLE 5.1. Let $\varepsilon(t) = [t]$ be the integer part of t . Then

$$\xi(t) = \sum_{k=1}^{\infty} N(t/k),$$

where $N(t)$ is the Poisson process. For every $t \geq 0$, the series consists of a finite number of summands and so converges almost surely. Note that $\xi(t)$ is not integrable for $t > 0$. The jump sizes of ξ are always one, and the jump times form a point process on \mathbb{R}_+ obtained as the superposition of the set of natural numbers scaled by $\Gamma_i, i \geq 1$.

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REFERENCES

- [1] M. Barczy, P. Kern, and G. Pap, *Dilatively stable stochastic processes and aggregate similarity*, Aequationes Math. 89 (6) (2015), pp. 1485–1507.
- [2] A. Basse-O'Connor and J. Rosiński, *On the uniform convergence of random series in Skorohod space and representations of càdlàg infinitely divisible processes*, Ann. Probab. 41 (6) (2013), pp. 4317–4341.
- [3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge 1989.
- [4] L. Chaumont and M. Yor, *Exercises in Probability: A Guided Tour from Measure Theory to Random Processes, via Conditioning*, second edition, Cambridge University Press, Cambridge 2012.
- [5] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods*, second edition, Springer, New York 2003.
- [6] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes. Volume II: General Theory and Structure*, second edition, Springer, New York 2008.
- [7] Y. Davydov, I. Molchanov, and S. Zuyev, *Strictly stable distributions on convex cones*, Electron. J. Probab. 13 (2008), paper no. 11, pp. 259–321.
- [8] K. Es-Sebaï and Y. Ouknine, *How rich is the class of processes which are infinitely divisible with respect to time?*, Statist. Probab. Lett. 78 (5) (2008), pp. 537–547.
- [9] S. N. Evans and I. Molchanov, *The semigroup of metric measure spaces and its infinitely divisible probability measures*, Trans. Amer. Math. Soc. 369 (3) (2017), pp. 1797–1834.
- [10] Ī. Ī. Gĭhman and A. V. Skorohod, *The Theory of Stochastic Processes. I*, Grundlehren Math. Wiss. 210, Springer, New York–Heidelberg 1974.
- [11] A. Hakassou and Y. Ouknine, *A note on α -IDT processes*, technical report, arXiv math:1207.0874, 2012.
- [12] A. Hakassou and Y. Ouknine, *IDT processes and associated Lévy processes with explicit constructions*, Stochastics 85 (6) (2013), pp. 1073–1111.
- [13] E. Iglói, *Dilative Stability*, Ph.D. thesis, University of Debrecen, Debrecen 2008.
- [14] E. Iglói and M. Barczy, *Path properties of dilatively stable processes and singularity of their distributions*, Stoch. Anal. Appl. 30 (5) (2012), pp. 831–848.
- [15] Z. Kabluchko and S. Stoev, *Stochastic integral representations and classification of sum- and max-infinitely divisible processes*, Bernoulli 22 (1) (2016), pp. 107–142.
- [16] I. Kaj, *Limiting fractal random processes in heavy-tailed systems*, in: *Fractals in Engineering: New Trends in Theory and Applications*, J. Lévy-Véhel and E. Lutton (Eds.), Springer, London 2005, pp. 199–217.
- [17] R. LePage, M. Woodroffe, and J. Zinn, *Convergence to a stable distribution via order statistics*, Ann. Probab. 9 (4) (1981), pp. 624–632.
- [18] R. Mansuy, *On processes which are infinitely divisible with respect to time*, arXiv math:0504408, 2005.
- [19] G. Maruyama, *Infinitely divisible processes*, Teor. Verojatn. Primen. 15 (1970), pp. 3–23.
- [20] I. Molchanov, *Theory of Random Sets*, Springer, London 2005.
- [21] M. D. Penrose, *Semi-min-stable processes*, Ann. Probab. 20 (3) (1992), pp. 1450–1463.
- [22] J. Rosiński, *On path properties of certain infinitely divisible processes*, Stochastic Process. Appl. 33 (1) (1989), pp. 73–87.

- [23] J. Rosiński, *On series representations of infinitely divisible random vectors*, Ann. Probab. 18 (1) (1990), pp. 405–430.
- [24] J. Rosiński, *Series representations of Lévy processes from the perspective of point processes*, in: *Lévy Processes*, O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick (Eds.), Birkhäuser, Boston, MA, 2001, pp. 401–415.
- [25] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York 1994.
- [26] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge 1999.
- [27] D. S. Silvestrov, *Limit Theorems for Randomly Stopped Stochastic Processes*, Springer, London 2004.

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